$\mathbb{F}_p^{alg}((t))$

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The object

Let p be a prime number.
$\mathbb{F}_p^{alg}$ the algebraic closure of the field with p-elements $\mathbb{F}_p$.
$\mathbb{F}_p^{alg}((t))$ field of Laurent series over $\mathbb{F}_p$, i.e. the completion of the field of rational functions $\mathbb{F}_p(t)$ with respect to the t-adic valuation.
Note that $\mathbb{F}_p^{alg}(t)^h$ is existentially closed in $\mathbb{F}_p^{alg}((t))$ (and in fact, after Rideau-Scanlon it is an elementary substructure)

Wanted result

Conjecture (D.F.O)

The below theory T of equicharacteristic valued fields (p, p) with a constant t in the field sort, given by the following description is complete and model complete:

$(K, v) \models T$ if and only if,

1. v is henselian,
2. the residue field K/v is algebraically closed,
3. the value group vK is a $\mathbb{Z}$-group,
4. $v(t) = \min vK_{>0}$,
5. $(1, t, \dots, t^{p-1})$ is a basis of K^p-vector space K,

VdD(t,n) For every 1-variable *monic separable additive* polynomial φ of degree p^n, with a unique *jump value*

$$\forall x \exists y \;\; x - \varphi(y) \in (K^{p^n})^{\perp}$$

where $(K^{p^n})^{\perp} = \oplus_{i=1}^{p^n-1} K^{p^n} t^i$.

Consequences

1. $\mathbb{F}_p^{alg}((t))$ is decidable.
2. $\mathbb{F}_p^{alg}(t)^h$ is the prime model of T.

Motivation

1. Papers by Kuhlmann, van den Dries, Anscombe (especially on optimal approximation), Kuhlmann's incompleteness result of the naive axiomatization of $\mathbb{F}_p((t))$.
2. Some early discussions (around 2012) with Salih Durhan.
3. Module theory of valued difference fields (O.).

First (positive-) observations about T

1. Every model of T is defectless.
2. Assuming resolution of singularities $\mathbb{F}_p^{alg}(t)^h$ is existentially closed in every model.
3. Image of every (possibly multivariate) additive polynomial is $\forall$-definable.
4. If $(L \subseteq K, v)$ a pair of models then $L/\mathcal{O}_L$ is existentially closed in $K/\mathcal{O}_K$ as a $F[Frob]$-module, where $\mathbb{F}_p \subseteq F \subseteq \mathbb{F}_p^{alg}(t)^h$
5. $K \models T$. If $\mathcal{O}_w(K) \supseteq \mathcal{O}_v(K)$ then $\mathcal{O}_w(K) \models VdD(t, n)$ and $K/w \models VdD(t, n)$.
6. If $[L \models T - VdD(t, n)]$ and w_0 is the finest proper coarsening of v then L/w_0 and $\mathcal{O}_{w_0}(L) \models VdD(n)$.
7. $K \models T$ and K'/K finite separable with a uniformer u. Then $K' \models VdD(u, n)$.

Observation about possible obstacles in a proof

1. $T-$ VdD(n) + "v defectless" is not complete.
2. There is $L \models (T- \ VdD(n,t))$ + "v is defectless", such that vL is not stably embedded.
3. K^{p^∞} satisfies Kaplansky conditions but, when K is ω-saturated K^{p^∞} is **never** algebraically maximal since we have
$$K = K^{p^\infty} + \varphi(K)$$
where $\varphi(x) = x^p - x$ (:true for any ω-saturated characteristic p field).
4. We can have $L \models T - VdD(n,t)$, $L \subset L' \subset K$, $K \models T$ such that L'/L immediate algebraic of degree p, but non-Galois.

About a proof

Let $k \subset K$ both models of T (K being already quite big).
We want to embed K over k, into a big elementary extension, say U, of k.

Proposition

Let $L \subset K$, be a maximal one, among the subfields of K which are embeddable in U over k. Then K/L is immediate.

(naive-)proof.

1. Use Hensel Lemma and the fact that $K^{p^\infty}/v = K/v$ for handling residual extensions.
2. Use that $K^{p^\infty}/v = K/v$ is infinite for value group extensions (and yes, we use the residue field for value group extensions) and use "paths".

$\square$

Why naive?

We want to show $L = K$. But naive proof only shows that

$$L \models T - VdD(n, t).$$

1. L can be not relatively algebraically closed in K.
2. There are examples $L \subseteq K$, L relatively alg. closed in K but L not existentially closed in K.

Example: We can have $\varphi(\alpha) := \alpha^p - \alpha = a \in L$. $L(\alpha)/L$ immediate
How to embed $L(\alpha)$ into U?

Idea: If $L/\mathcal{O}_L$ is existentially closed in $K/\mathcal{O}$ then we are done. Hence the idea is to "update" the proof of above proposition with adding the sort $K/\mathcal{O}$.
But this difficult (and we do not really need).

We need to deal with K^{p^∞}

Set $\varphi(x) := x^p - x$.

Proposition

Let α_γ be a p.c sequence such that $\varphi(a_\gamma) \rightsquigarrow a$ then there is β such that

1. $\alpha_\gamma \rightsquigarrow \beta$,
2. $\varphi(\beta) - a \in K^{p^\infty}$.

We have to be able to know (for example) $\varphi(K) \cap K^{p^\infty}$.

What is the power of $VdD(t, n)$

$K \models T$. Let $a \in K$. Write

$$a = \varphi(a_0) + a_1^p t + \dots a_{p-1}^p t^{p-1}$$

and

$$a = \varphi(b_0) + b_1^p t + \dots b_{p-1}^p t^{p-1}.$$

Then $a_0 - b_0 \in \mathcal{O}$ and $a_i - b_i \in \mathcal{O}$ for all i. Moreover

$$v(\varphi(a_0) - a) = \max_{x \in K}\{v(\varphi(x) - a)\} = \min_{i \geq 1} pv(a_i) + i.$$

(Same with b_0 and b_i's).
notation: $a_0/\mathcal{O} := a^{\varphi}$ and $a_i/\mathcal{O} := (a)^{\perp\varphi}{}_i$.

$(a)^{\perp\varphi}{}_i$ and $\lambda_i(a)$

The a_i's and $\lambda_i(a)$'s can be quite different:
$a \in K^{p^\infty} \setminus \varphi(K)$.
$\lambda_i(a) = 0$ for all $i \geq 1$ but $v(a_i) < 0$ for some i.
But we have

$$(a)^{\perp\varphi}{}_i = \lambda_i(a^\varphi)$$

for all $i \geq 1$.
This is also true for all additive ψ instead of φ with coefficients in K^{p^∞}.

Interlude: Non-standart Frobenius

?: How to make K^{p^∞} concrete.
Let $k \models T$. Take $K := k^{\mathbb{N}}/U$ for a non principal ultrafilter U over $\mathbb{N}$.
Let $\sigma[(x_i)_i] := [(x_i^{p^i})_i] (\mod U)$. Then $\sigma(K)^{1/p^\infty} \subset K^{p^\infty}$, an isomorphic copy of K^{1/p^∞}.

We have for x, $x = x^\sigma + \varphi(x_0)$ for some $x_0 \in K$.

Let $k_0 := \mathbb{F}_p^{alg}(t)^h$.
$k_1 := \sigma(k_0)^{1/p^\infty}(t)^h$, and $k_i = \sigma(k_{i-1})^{1/p^\infty}(t)^h$.
$k_\sigma := \bigcup_i k_i$.
Then k_σ goes up to a minimal model.

Proposition (almost)

Let $k \subseteq K$ both models of T, and U a big saturated elementary extension of k. Let $L \subset K$, be a maximal subfield among the subfields of K which are embeddable in U over k. Then K/L is immediate and $L \models T$.

Thank you.